CMPT 478/981 Spring 2025 Quantum Circuits & Compilation

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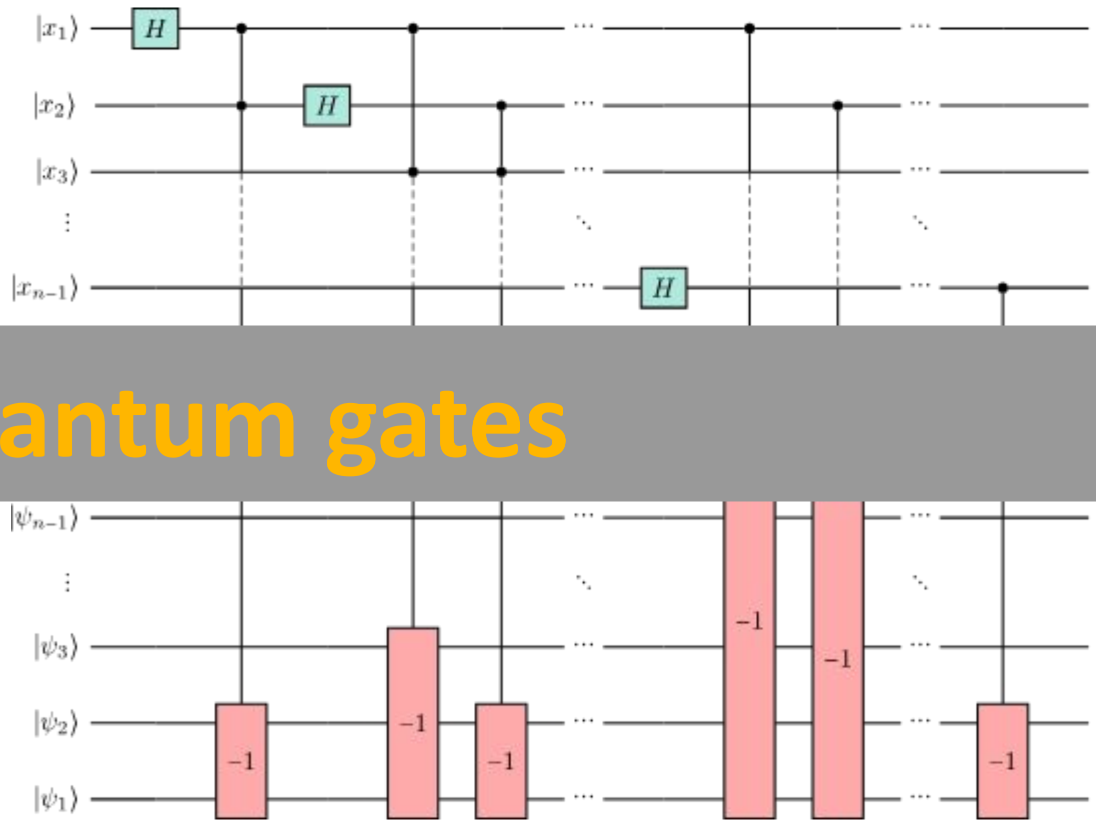
Today’s agenda

- Gates & Resources
  - Gate constructions
  - Universality
  - Physical & logical gates
  - Computational bottlenecks at the physical & logical layer

Quantum gates



Quantum universality

- Quantum computing has two notions of a universal gate set

- Exact universality (hardware layer):

  *A gate set G is **exactly** universal if every 2^n by 2^n unitary matrix can be implemented by a circuit over G*

- Approximate universality (logical layer):

  *A gate set G is **approximately** universal if every unitary matrix can be implemented by a circuit over G up to error e in the operator norm*

$$||U - \tilde{U}|| = \max_{|\psi\rangle} ||(U - \tilde{U})|\psi\rangle|| \leq e$$

Exact universality

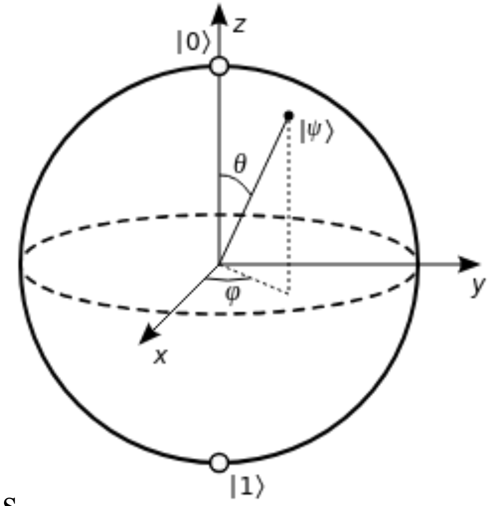
- Generic form:
  - Any entangling gate (e.g. CNOT, CZ) + single qubit unitaries
- A little bit more specific
  - CNOT + Z & Y (or X) rotations
- Even more specific
  - CZ + fixed axis rotations
  - ZZ interactions + fixed axis rotations
  - Molmer-Sorenson + fixed axis rotations
  - Beamsplitters + phase shifters + entangled states

More specialized

Single-qubit unitaries

Notation:
U(n) - n x n unitary matrices
SU(n) - Determinant 1 subgroup of U(n)
O(n) - n x n orthogonal matrices
SO(n) - Determinant 1 subgroup of O(n)

- "Rotation gates"
- Formally…
  - Equate unitaries up to global phase
  - $|\mathrm{Det}(U)| = 1 \Rightarrow U(2) \simeq SU(2) \times U(1)$
  - $SU(2) \simeq SO(3)$
  - $U(2) \simeq SO(3) \times U(1)$
- How to implement an arbitrary rotation in SO(3)?
  - $\Rightarrow$ Euler angles!
  - Informally, every rotation in 3-space is a product of 3 rotations along non-parallel axes, e.g. x-z-x



Fixed axis rotations

- Exponentials of Pauli gates give rise to rotations around the X, Y, and Z axes

$$R_X(\theta) = e^{-i\frac{\theta}{2}X} = \begin{bmatrix} \cos\theta/2 & i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

$$R_Y(\theta) = e^{-i\frac{\theta}{2}Y} = \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

$$R_Z(\theta) = e^{-i\frac{\theta}{2}Z} = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

Matrix exponentials

- Matrix exponential defined by the Taylor series expansion

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

- Example: diagonal case

- Example: diagonalizable ($X=U\Lambda U^\dagger$) case

Spectral theorem

- A matrix X is diagonalizable if and only if it is normal
- In this case, $X = U\Lambda U^\dagger$ where
  - The entries of Λ are the eigenvalues of X
  - The columns of U are associated eigenvectors
- Example: X

Single qubit universality

Theorem: Any 1-qubit unitary U can be decomposed up to global phase as

$$R_Z(a)R_Y(b)R_Z(c)$$

Also works for any 2 non-parallel axes

Multi-qubit gates

- At least one (entangling) multi-qubit gate needed for universality
- Turns out that suffices (more on this later…)

Clifford

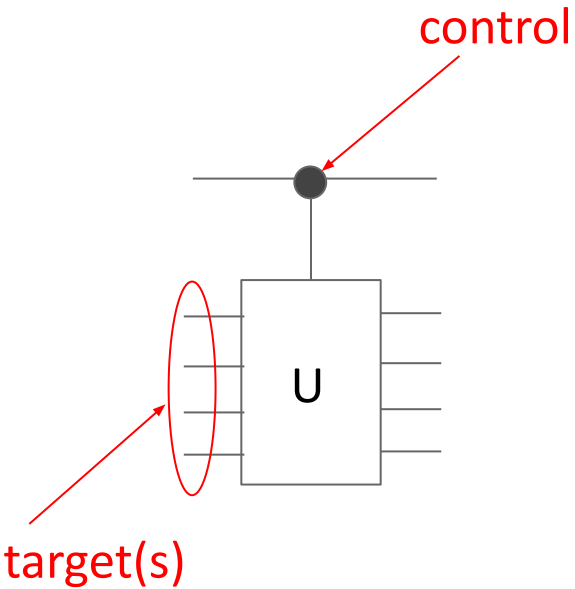
$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

non-entangling

$$CS = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix} \quad TOF = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad CCZ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

non-Clifford

Controlled gates



Informally:

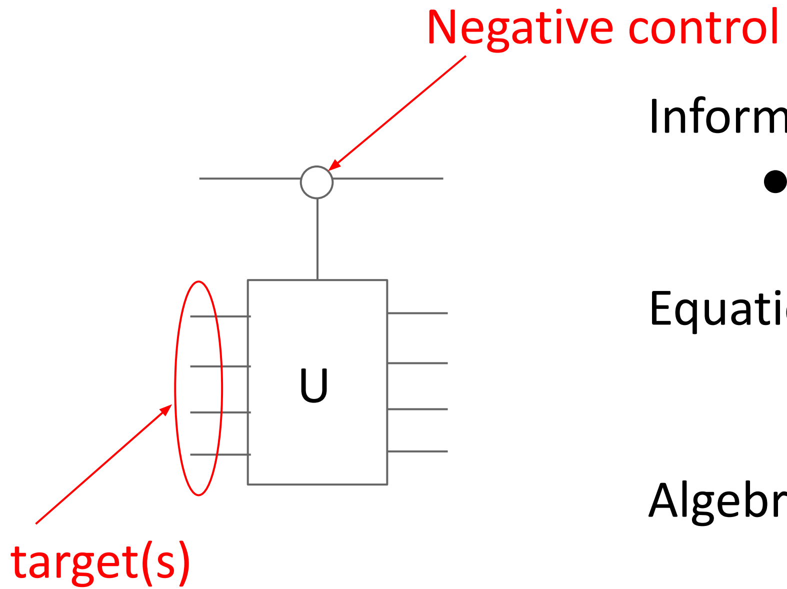
- Applies U conditional on the control qubit in the |1> state

Equationally:

Algebraically:

CNOT & CZ: a love story

(negative) Controlled gates



Informally:

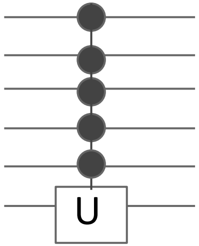
- Applies U conditional on the control qubit in the |1> state

Equationally:

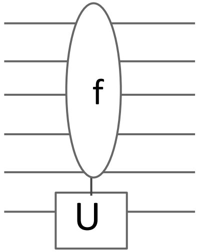
Algebraically:

Multiply-controlled gates

- Can cascade controls: $C^kU = C(C\ldots(CU)\ldots)$
- Informally, applies if all controls are $|1\rangle$



- More generally, can have single-target gates which apply U if and only if some control function $f: \{0,1\}^k \to \{0,1\}$ evaluates to 1

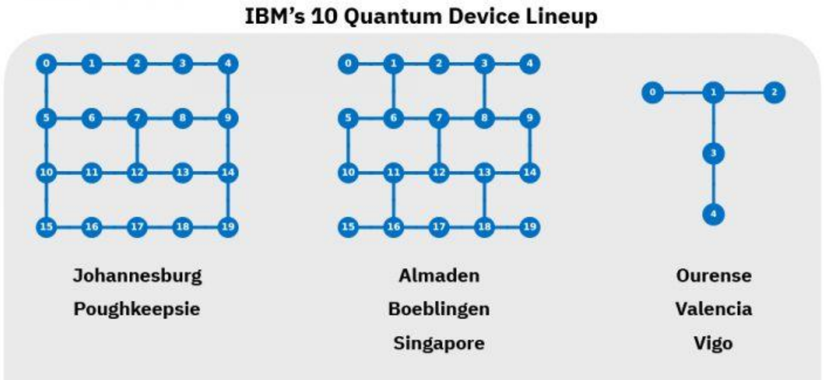


Multi-qubit exponentials

- Some physical devices have “Z-Z” or “X-Y” or “X-X-X-X-...-X” interactions
- Generally, this means a tunable multi-qubit gate, e.g.

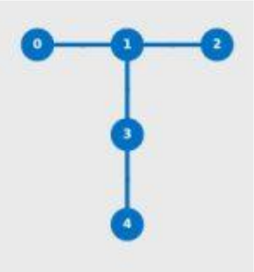
$$e^{i\frac{\theta}{2}Z\otimes Z}$$

- Whether tunable or not, physical devices typically have restricted connectivity, in that only certain pairs of qubits can interact



Circuit routing/mapping: compiling to conform to the hardware topology

Example: circuit routing



A note on physical fidelities

- Physical compilation seeks to **increase** fidelity

- Single-qubit gates have **high** fidelity
  - Around of 99.9% recently
- Multi-qubit interactions have **low** fidelity
  - Getting better, but still generally under 99%

- Ways to increase fidelity
  - Reduce CNOT/multi-qubit gate counts
  - Reduce depth
  - Reduce crosstalk

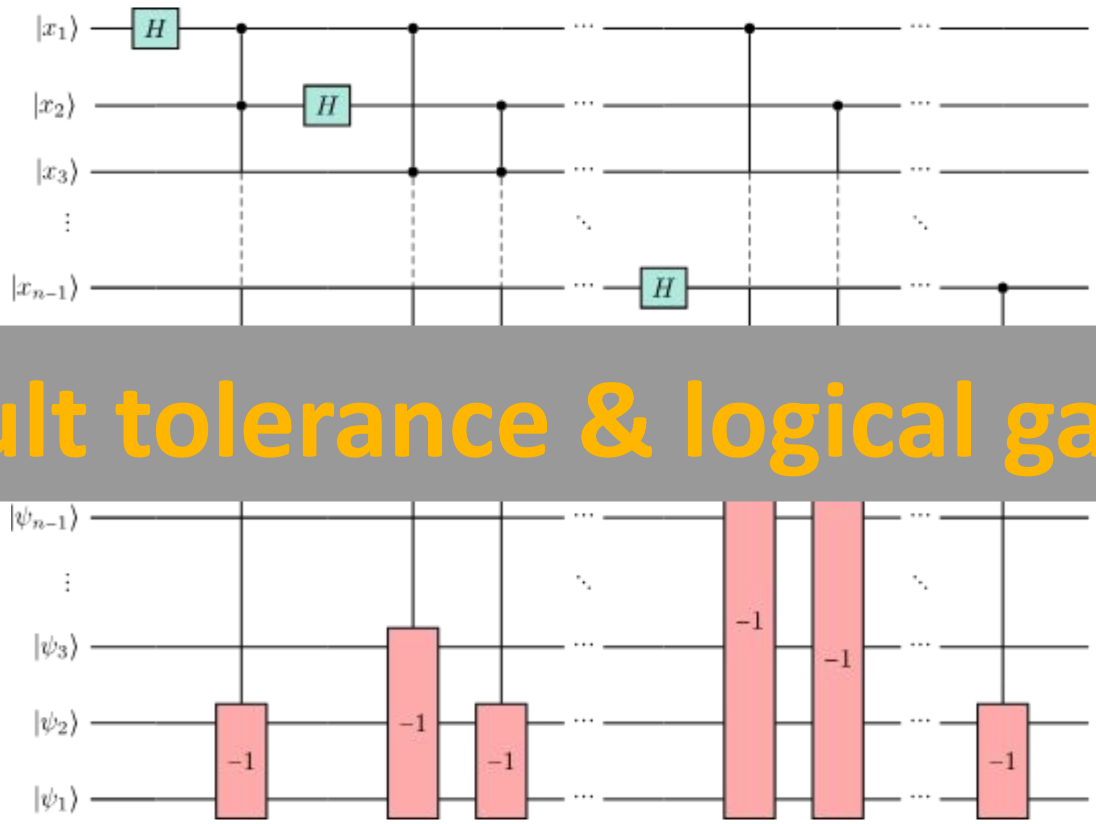
Task for physical compilation is usually to route the circuit with the highest fidelity (e.g. fewest & shallowest CNOTs)

Rigetti, circa 2019

T_1	T_2^*	$\mathcal{F}_{1q}$	$\mathcal{F}_{RO}$
μs	μs		
15.2 $\pm$ 2.5	**7.2** $\pm$ 0.7	0.9815	0.938
17.6 $\pm$ 1.7	**7.7** $\pm$ 1.4	0.9907	0.958
18.2 $\pm$ 1.1	**10.8** $\pm$ 0.6	0.9813	0.970
31.0 $\pm$ 2.6	**16.8** $\pm$ 0.8	0.9908	0.886
23.0 $\pm$ 0.5	**5.2** $\pm$ 0.2	0.9887	0.953
22.2 $\pm$ 2.1	**11.1** $\pm$ 1.0	0.9645	0.965
26.8 $\pm$ 2.5	**26.8** $\pm$ 2.5	0.9905	0.840
29.4 $\pm$ 3.8	**13.0** $\pm$ 1.2	0.9916	0.925
24.5 $\pm$ 2.8	**13.8** $\pm$ 0.4	0.9869	0.947
20.8 $\pm$ 6.2	**11.1** $\pm$ 0.7	0.9934	0.927
17.1 $\pm$ 1.2	**10.6** $\pm$ 0.5	0.9916	0.942
16.9 $\pm$ 2.0	**4.9** $\pm$ 1.0	0.9901	0.900
8.2 $\pm$ 0.9	**10.9** $\pm$ 1.4	0.9902	0.942
18.7 $\pm$ 2.0	**12.7** $\pm$ 0.4	0.9933	0.921
13.9 $\pm$ 2.2	**9.4** $\pm$ 0.7	0.9916	0.947
20.8 $\pm$ 3.1	**7.3** $\pm$ 0.4	0.9852	0.970
16.7 $\pm$ 1.2	**7.5** $\pm$ 0.5	0.9906	0.948
24.0 $\pm$ 4.2	**8.4** $\pm$ 0.4	0.9895	0.921
16.9 $\pm$ 2.9	**12.9** $\pm$ 1.3	0.9496	0.930
24.7 $\pm$ 2.8	**9.8** $\pm$ 0.8	0.9942	0.930

	A_0	f_m	t_{CZ}	$\mathcal{F}_{2q}$
	Φ/Φ_0	MHz	ns	
0 – 5	0.27	94.5	168	0.936
0 – 6	0.36	123.9	197	0.889
1 – 6	0.37	137.1	173	0.888
1 – 7	0.59	137.9	179	0.919
2 – 7	0.62	87.4	160	0.817
2 – 8	0.23	55.6	189	0.906
4 – 9	0.43	183.6	122	0.854
5 – 10	0.60	152.9	145	0.870
6 – 11	0.38	142.4	180	0.838
7 – 12	0.60	241.9	214	0.870
8 – 13	0.40	152.0	185	0.881
9 – 14	0.62	130.8	139	0.872
10 – 15	0.53	142.1	154	0.854
10 – 16	0.43	170.3	180	0.838
11 – 16	0.38	160.6	155	0.891
11 – 17	0.29	85.7	207	0.844
12 – 17	0.36	177.1	184	0.876
12 – 18	0.28	113.9	203	0.886
13 – 18	0.24	66.2	152	0.936
13 – 19	0.62	109.6	181	0.921
14 – 19	0.59	188.1	142	0.797

Fault tolerance & logical gates



Error correction & fault tolerance

- Error rate of a classical transistor/gate/operation: 10^{-17}
- Error rate of a quantum gate: $10^{-2} – 10^{-3}$ optimistically
  - Means computations can only run ~100 gates without error correction

What do we do?

- Option 1: Use QCs to prepare a large, shallow state & measure
  - Eg. Variational quantum eigensolver, Quantum approximate optimization, Quantum ML
  - Measurement-based QC kind of fits in here depending on implementation
- Option 2: Compute fault-tolerantly (FT QEC)
  - Encode our state with an error correcting code (quantum error correction)
  - Perform computation directly on the encoded state (Fault-tolerance)

3-bit repetition code

Encoded gates

- “Umbrella” argument ⇒

- To solve it, apply gates directly on the encoded data

Error propagation

- For fault-tolerance, gates must also not propagate errors

Fundamental theorem of FTQEC

(not really)

Theorem (Eastin-Knill):

For any non-trivial QECC, there is no (finite or infinite) set of universal, transversal encoded gates

Non-transversal gate constructions

- Typically based on gate teleportation and magic state distillation

Relative cost of FT implementations

Transversal:

- Error rate: $\propto p_{hardware}$
- Circuit cost: O(1)

Gate teleportation:

- Error rate: $\propto p_{hardware} + p_{MSD}$
- Circuit cost: $O(1) + O(1/p_{MSD})$

Universality at the logical layer

So, we need a gate set which:

1. Is universal, and
2. Is efficient, and
3. Can be implemented fault-tolerantly on a code, via
   - Transversal (**efficient**) implementations
   - Gate teleportation/code switching/etc. (**inefficient**) implementations

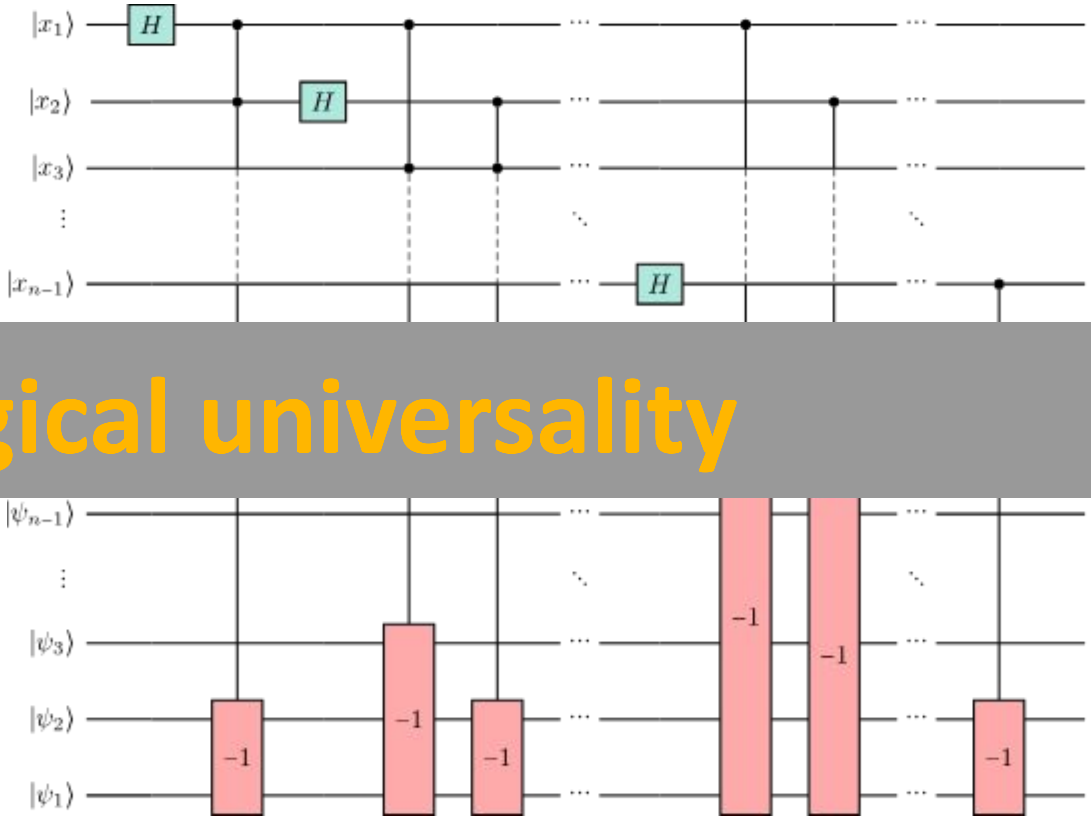
The need for approximation

- Single-qubit unitaries are uncountable
  - $\Rightarrow$ No finite or even **countable** gate set can implement **all** 1-qubit unitaries
  - Would give us extraordinary power (all Turing degrees) with just 1 qubit
- Physically, no real problem
  - Start with some floating point approximation θ in code
  - Control system applies pulse along y direction for t nanoseconds in order to rotate by θ
  - Check fidelity via tomography during tuning & fold the approximation into the error rate
- At the logical layer, we don't have that freedom
  - Have to settle for approximations

Logical compilation

- General scheme:
  - Compile to multi-qubit gate of choice + single qubit rotations
  - Approximate single qubit rotations
- Errors are subadditive:

Logical universality



The Pauli group & QECC's

Pauli group:

Stabilizer codes:

The Clifford group

Clifford simulation

Theorem (Gottesman-Knill):

Any circuit consisting of Clifford operations & computational basis measurements is classically simulable in polynomial-time (in n)

Measurement of Cliffords

Clifford group and transversality

See a QECC course…

Theorem:

The Clifford group can be performed transversally in any self-dual CSS code

Implications:

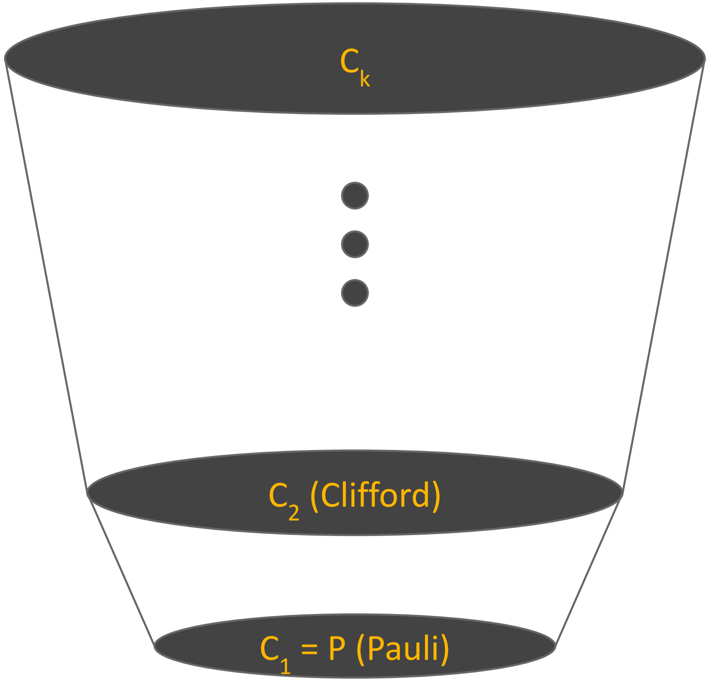
1. In most codes, we can do Clifford group efficiently
2. To achieve universality, need an inefficient non-Clifford gate

A steady source of non-Clifford gates: The Clifford hierarchy

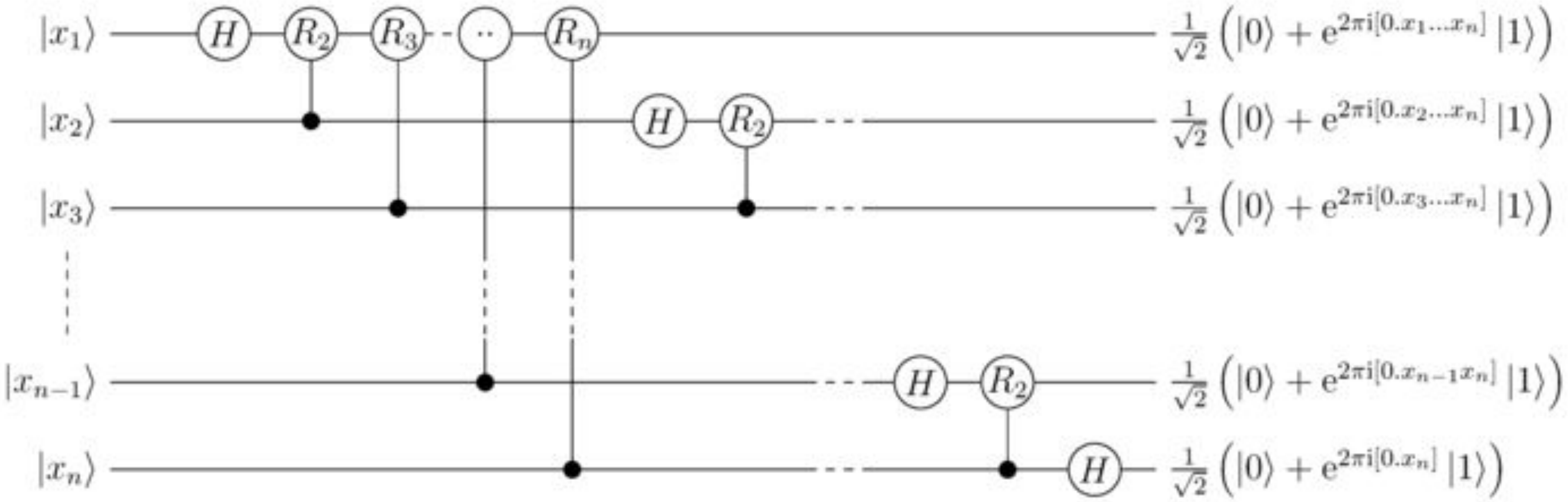
- Clifford hierarchy is defined as

$$\mathcal{C}_{1,n} = \mathcal{P}_n$$
$$\mathcal{C}_{k,n} = \{U\mathcal{P}_nU^\dagger \subseteq \mathcal{C}_{k-1,n} \mid U \in U(2^n)\}$$

- Important since they can be implemented via gate teleportation



The QFT is teleportable



$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^k} \end{bmatrix} \in \mathcal{C}_k$$

The T gate

- Clifford+anything is (approximately) universal, so what should we choose?
- All hail the holy third-level T gate, an order 8 Z-axis rotation:

$$T = R_3 = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}, \qquad \omega = e^{i\pi/4}$$

- Admits a simple (and efficient) gate teleportation scheme:

- Magic state distillation on the other hand, not so nice:
  - 15-to-1 with cubic error suppression

Approximate universality of Clifford+T

- Fact: **HTHT & THTH** are rotations of irrational multiples of pi around non-parallel axes

Efficiency of approximation

- **Solovay-Kitaev theorem (~1995)**
  Given an inverse-closed set of single-qubit unitaries, any single-qubit Unitary can be approximated to accuracy e using $O(\log^c(1/e))$ gates

- **Ross-Selinger (2014)** gets it down to $3\log(1/e) + O(\log\log(1/e))$ via *ring round-off & number-theoretic characterization*
  We'll talk a bit about this one

- **Bocharov-Roetteler-Svore (2015)** down to $\log(1/e) + O(\log\log(1/e))$ expected cost via probabilistic techniques

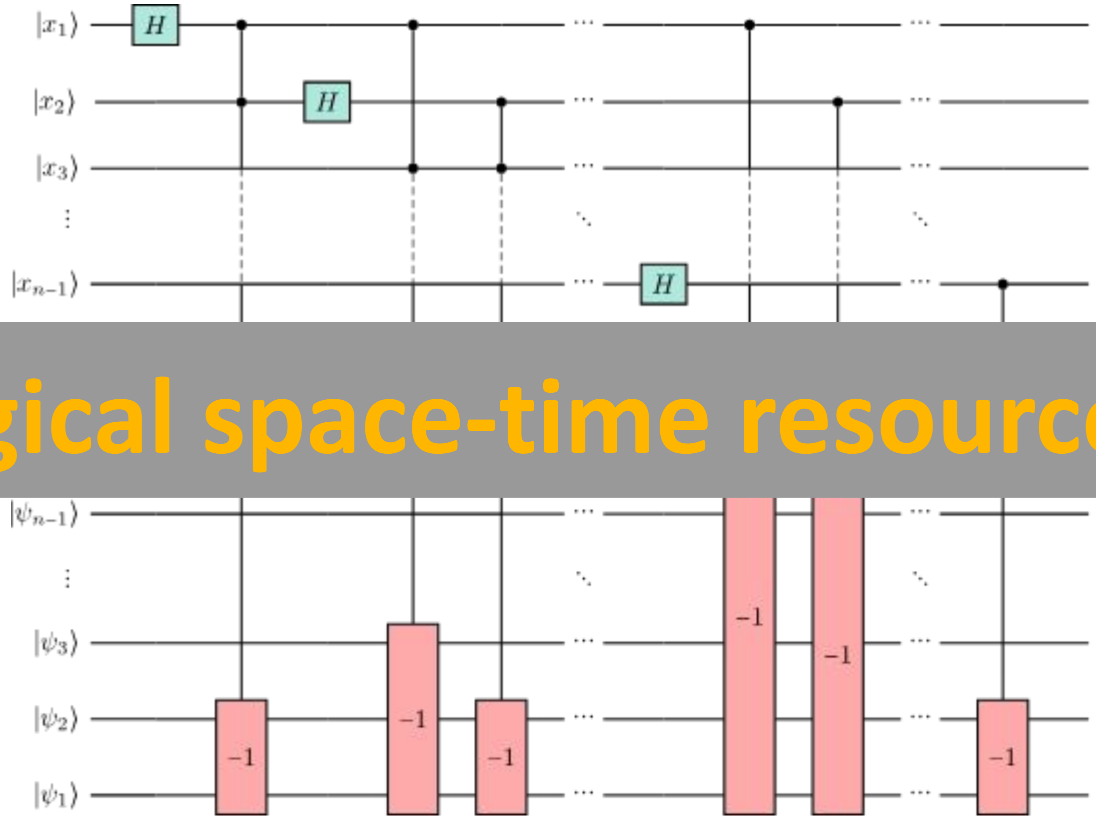
Clifford+T*

We know:

- Clifford+T is (approximately + efficiently) universal
- It can be implemented in most codes
- The T gate is (theoretically) more expensive than Clifford gates

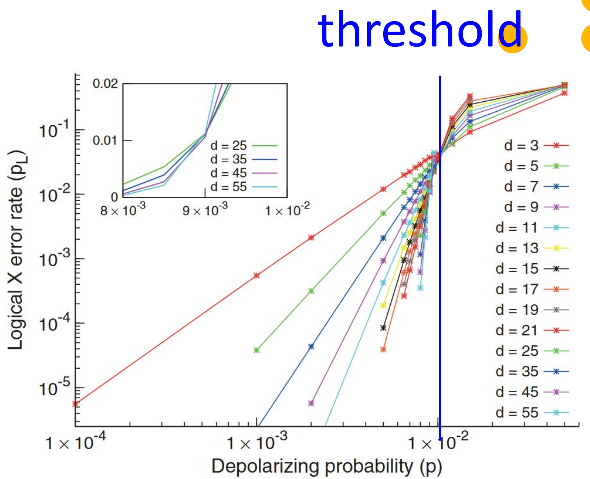
To ensure we're working under correct assumptions, need to keep re-visiting cost of FT implementations!

*Funny story: the term was coined in my research group back in 2011



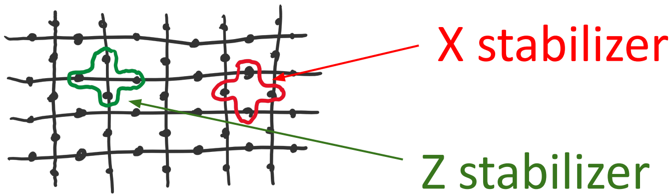
Logical space-time resources

The surface code

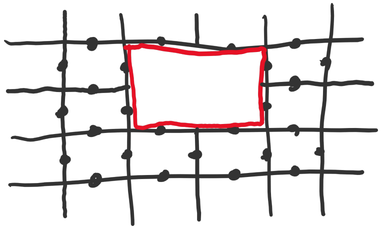


- Based on Kitaev's toric code
- Since 2010's, most promising candidate for FTQEC
  - Threshold around 10^{-2} vs 10^{-5} for Steane code
  - Can be implemented on a 2D lattice ("low density")

- Define two types of stabilizers on a 2D lattice:

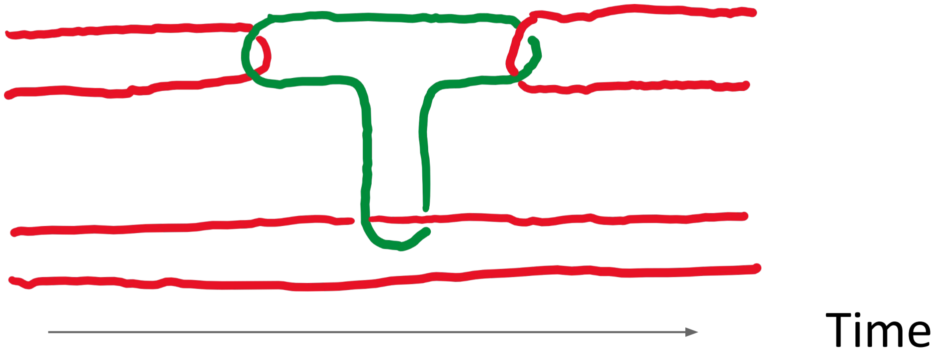


- "Turn off" stabilizers in a section (a defect) to make a qubit:

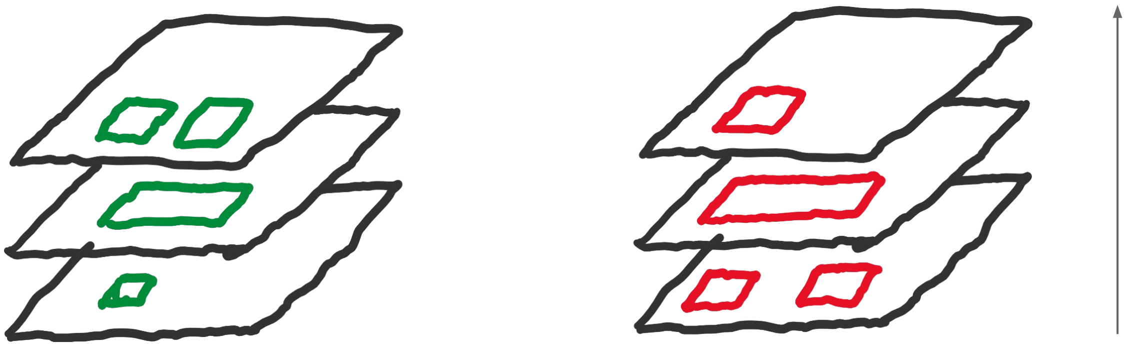


Fault tolerant (Clifford) gates in the surface code

- Circa 2010's: Braiding

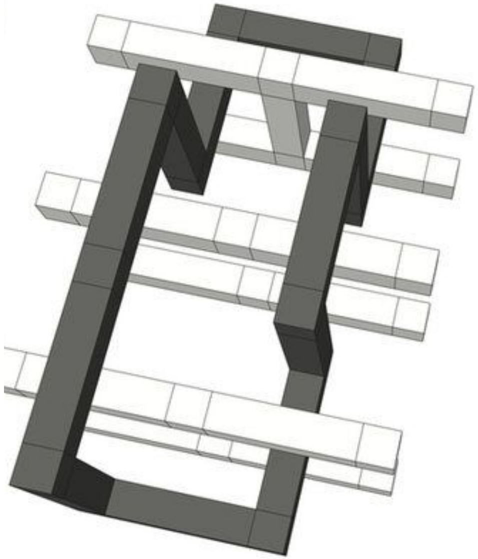


- Now: Lattice surgery

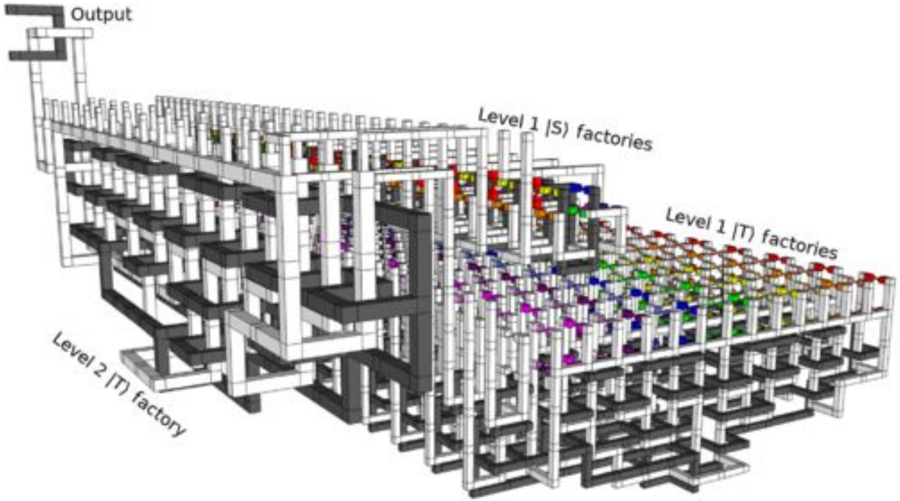


Relative space-time volumes

CNOT:

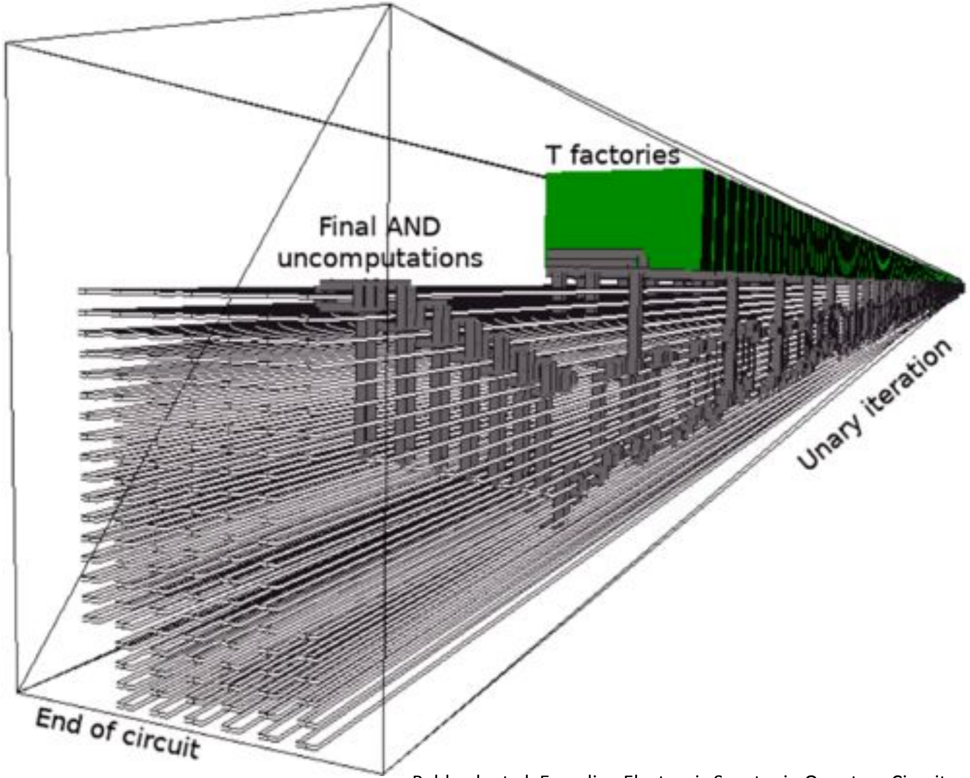


T distillation factory:



Babbush et al. Encoding Electronic Spectra in Quantum Circuits with Linear T Complexity. Phys. Rev. X 2018.

A compiled FTQEC computation



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Lattice surgery

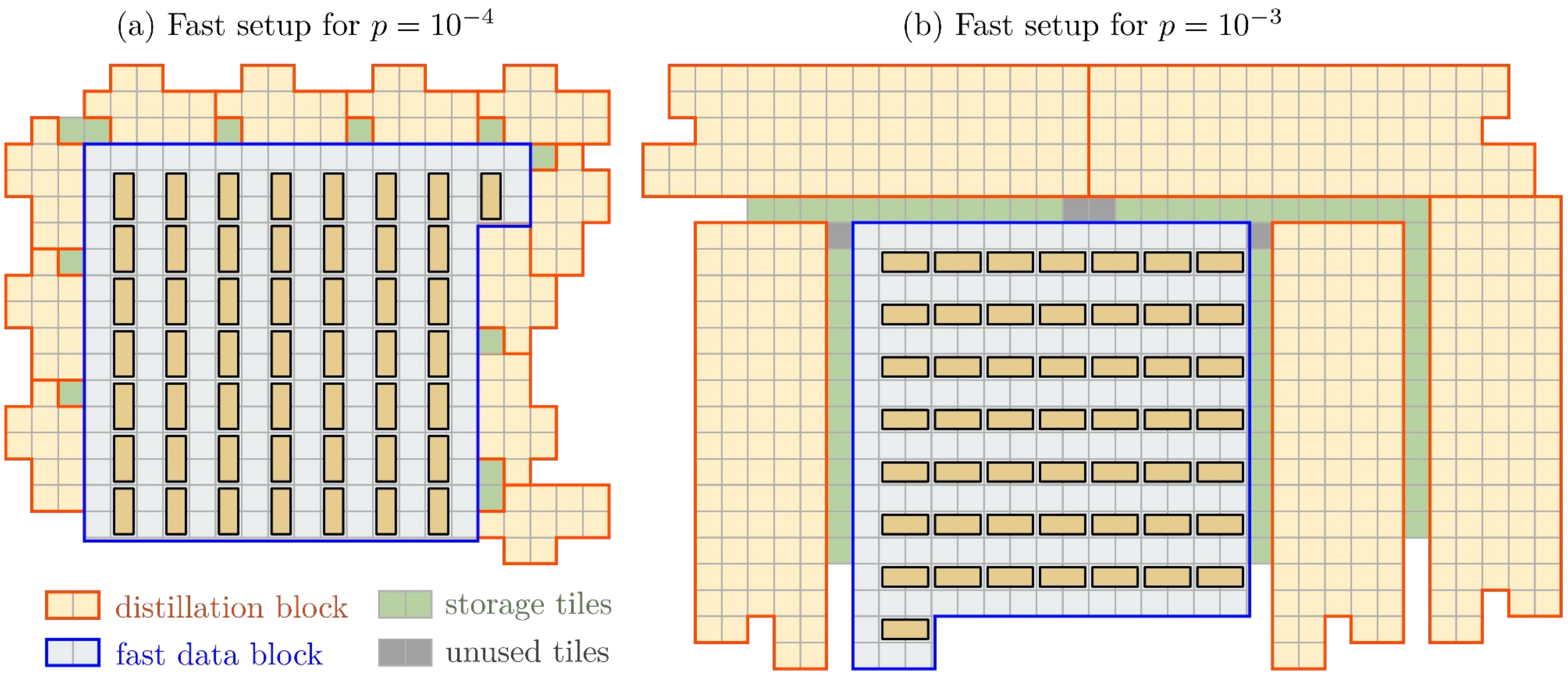


Figure 23: Fast setups using fast data blocks and 11 15-to-1 distillation blocks for $p = 10^{-4}$ or 5 116-to-12 distillation block for $p = 10^{-3}$.

Litinski, A Game of Surface Codes, Quantum 2019.

Maybe not…

arXiv > quant-ph > arXiv:1905.06903

Quantum Physics

[Submitted on 16 May 2019 (v1), last revised 6 Nov 2019 (this version, v3)]

Magic State Distillation: Not as Costly as You Think

Daniel Litinski

arXiv > quant-ph > arXiv:2409.17595v1

Quantum Physics

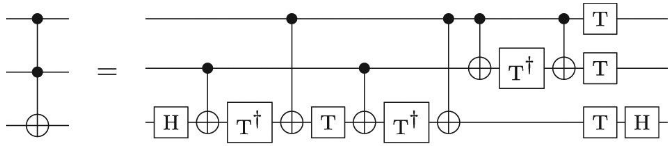
[Submitted on 26 Sep 2024]

Magic state cultivation: growing T states as cheap as CNOT gates

Craig Gidney, Noah Shutty, Cody Jones

What about other non-Clifford gates?

- Toffoli+Hadamard is also universal
  - …but the Toffoli gate is best implemented by using 7 T gates (**optimal**) in most cases



- What about gates from higher levels?
  - …relies on |T> states to implement via gate teleportation
  - …but can result in more efficient impl’s in some regimes

arXiv > quant-ph > arXiv:1603.04230

Quantum Physics

[Submitted on 14 Mar 2016 (v1), last revised 14 Oct 2016 (this version, v2)]

An efficient magic state approach to small angle rotations

Earl T. Campbell, Joe O'Gorman

Recap

- Basic gate constructions
  - Rotations
  - Controlled gates
  - (Pauli) Exponentials
- Universal gate sets
  - “Single qubit + entangling”
  - “Clifford + one non-Clifford gate”
- Computational bottlenecks
  - Physical: entangling gates
  - Logical: T gates (or non-Cliffords)

Next class: compilation